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Semi-Lagrangian Schemes for Hamilton–Jacobi Equations, Discrete Representation Formulae and Godunov Methods[1]

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We study a class of semi-Lagrangian schemes which can be interpreted as a discrete version of the Hopf–Lax–Oleinik representation formula for the exact viscosity solution of first order evolutive Hamilton–Jacobi equations. That interpretation shows that the scheme is potentially accurate to any prescribed order. We discuss how the method can be implemented for convex and coercive Hamiltonians with a particular structure and how this method can be coupled with a discrete Legendre trasform. We also show that in one dimension, the first-order semi-Lagrangian scheme coincides with the integration of the Godunov scheme for the corresponding conservation laws. Several test illustrate the main features of semi-Lagrangian schemes for evolutive Hamilton–Jacobi equations.

Key Words: semi-Lagrangian schemes; Hamilton–Jacobi equations; Godunov methods; representation formulae.

1. INTRODUCTION

We deal with a class of semi-Lagrangian schemes for evolutive Hamilton–Jacobi equation of the first order. In particular, we consider the model problem

$$\begin{aligned} v_t + H(\nabla v) &= 0, \qquad \text{in } \mathbb{R}^N \times \mathbb{R} \\ v(x,0) &= v_0(x) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{1.1}$$

Many approximation schemes have been proposed since the paper by Kružkov [26], the most popular schemes are based on finite differences. Crandall and Lions [14] have shown

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that monotone schemes for (1.1) are at most first order. More recently other methods have been proposed extending to Hamilton–Jacobi equations high-order methods for conservation laws in order to avoid the intrinsic limitations of monotone schemes. Among those contributions we quote the paper by Osher and Shu [32] where the technique of ENO (Essentially Non-Oscillatory) schemes has been applied to Hamilton–Jacobi equation for the first time and the more recent contributions by Jiang and Peng [25] and by Lin and Tadmor [28].

As we said, we focus our attention on a class of semi-Lagrangian (SL) methods for (1.1). The first semi-Lagrangian method for conservation laws has been proposed by Courant–Isaacson–Rees in [13]. Since then many other problems mainly related to fluid dynamics and meteorological applications have been solved using SL-schemes; see, e.g., [34]. As far as Hamilton–Jacobi equations are concerned, similar methods have been first applied to stationary Hamilton–Jacobi–Bellman equations related to optimal control problems; see e.g., [16, 17, 19]. In the control framework, a semi-Lagrangian scheme is obtained by discretizing in time the dynamic programming principle and this provides an interesting interpretation of the schemes in terms of a discrete representation formula for the value function (see [3] and [18] for more details and additional references). The same approach was used in [21] to solve evolutive problems with convex hamiltonians producing a first-order scheme. High-order schemes of the same type for the pure advection problem in $\mathbb{R}^N$ have been studied in [20], which contains quite an extensive analysis of their stability and convergence properties. Just to summarize, the above mentioned SL-schemes can compute the solution on unstructured as well as on structured grids, allow to use large time steps (at least larger than those allowed by finite differences schemes), and may compute high-order accurate approximations.

Although the theory has been mainly developed for first-order Hamilton–Jacobi equations, an extension to second-order problems is also possible (see [9, 10, 22, 35]).

In this paper we deal with SL-schemes related to the approximation of (1.1) which includes the first-order model equation for tracking the evolution of an interface by the "level set" method (see [31] and [33]). We will show how SL-schemes are strictly connected with the Hopf representation formula for the exact solution of (1.1). This connection is important for two main reasons. First, it shows that SL-schemes can produce arbitrarily accurate approximations for particular classes of problems. Second, one can regard the SL scheme for (1.1) as the analogue of Godunov scheme for conservation laws.

The paper is organized as follows. Section 2 contains the basic informations regarding the Hopf formula for the exact solution of (1.1) and shows how the formula can be actually computed for general convex Hamiltonians in $\mathbb{R}^N$. In Section 3 we prove that in $\mathbb{R}$ the SL-scheme coincides with the integration of the solution obtained by the Godunov method for the corresponding conservation laws. In Section 4 we present some properties of our schemes. Finally, Section 5 is devoted to numerical experiments.

2. CONTINUOUS AND DISCRETE REPRESENTATION FORMULAE

A crucial role in the representation formula for (1.1) is played by the Legendre–Fenchel conjugate of convex analysis which we recall here for reader's convenience.

DEFINITION 2.1. Let $H : \mathbb{R}^N \to \mathbb{R}$ be a continuous and convex function such that

$$\frac{H(p)}{|p|} \to +\infty \quad \text{for } |p| \to +\infty. \tag{2.1}$$

The Legendre–Fenchel conjugate of H is the continuous and convex function, H^*, defined by

$$H^*(p) \equiv \sup_{q \in \mathbb{R}^N} \{p \times q - H(q)\}. \tag{2.2}$$

It is worth noting that (2.1) guarantees that $H^*(p)$ is always properly defined and $(H^*(p))^* = H(p)$ for any $p \in \mathbb{R}^N$. However, as we will see later in this section, when (2.1) is not satisfied one can still compute H^* which will assume real values only on a subset of $\mathbb{R}^N$.

As we will see immediately, the Legendre–Fenchel conjugate is crucial in establishing a link between the general Cauchy problem (1.1) and a control problem. Through this link we obtain the representation formula for the exact solution.

If the Hamiltonian H satisfies the assumptions required in Definition (2.1), then we can write Eq. (1.1) as

$$v_t + \sup_{a \in \mathbb{R}^N} \{\nabla v \times a - H^*(a)\} = 0. \tag{2.3}$$

It is interesting to note (see [3], Ch. III for details) that the above equation is the Bellman equation for a finite horizon control problem with the controls varying in $A \equiv \mathbb{R}^N$, the controlled dynamics equal to

$$\dot{y}(t) = -a(t), \quad y(0) = x, \tag{2.4}$$

and the running cost equal to $H^*(a)$. We will denote by $y_x(t)$ the solution trajectory of (2.4) evaluated at time t. Obviously, it will depend on the choice of the control $a(\cdot)$.

It is also well known that the unique viscosity solution of the Cauchy problem (1.1) is the value function of the above control problem, i.e.,

$$v(x, t) = \inf_{a(\cdot) \in \mathcal{A}} \left[\int_0^t H^*(a(s))\, ds + v_0(y_x(t)) \right], \tag{2.5}$$

where

$$\mathcal{A} \equiv \{a(\cdot) : [0, T] \to A, \text{measurable}\}.$$

Since the optimal controls (i.e., the $a^*(\cdot)$ minimizing the right-hand side of (2.5)) are constant in time, the optimal trajectories are straight lines and the optimal trajectory y^* starting at the point x at time 0 is

$$y^*(t) = x - a^* t.$$

Then, substituting in (2.5) we obtain the Hopf–Lax–Oleinik representation formula

$$v(x, t) = \inf_{y \in \mathbb{R}^N} \left[v_0(y) + t H^* \left(\frac{x - y}{t} \right) \right]. \tag{2.6}$$

Later in this section we will discuss some extensions of the above representation formula to more general Hamiltonians.

Let us examine now the typical SL-scheme. In order to simplify notations we will consider $N = 2$, the extension to higher dimensions being straightforward.

Let us define the lattice $L(\Delta x, \Delta y, \Delta t)$ by

$$L \equiv \{(x_i, y_j, t_n) : x_i = i\Delta x, y_j = j\Delta y \text{ and } t_n = n\Delta t, \text{ for } i, j \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}, \quad (2.7)$$

where $(x_i, y_j, t_n) \in \mathbb{R}^2 \times \mathbb{R}_+$, Δx and Δy are the space steps, and Δt is the time step. In order to obtain the SL-scheme, let us consider the following approximation

$$-\nabla v(x_i, y_j, t_n) \cdot a = \frac{v(x_i - a_1\Delta t, y_j - a_2\Delta t, t_n) - v(x_i, y_j, t_n)}{\Delta t} + O(\Delta t). \quad (2.8)$$

We will use the standard notation $v^n_{i,j}$ for an approximation of $v(x_i, y_j, t_n)$, $i, j \in \mathbb{Z}$, and $n \in \mathbb{N}$ and $v^n : \mathbb{R}^2 \to \mathbb{R}$ for its reconstruction, i.e., its extension to any triple (x, y, t_n) (see details below). Replacing in (2.3) the term v_t by forward finite differences and the directional derivative by (2.8), we get

$$\frac{v^{n+1}_{i,j} - v^n_{i,j}}{\Delta t} = \min_{a \in \mathbb{R}^2}\left[\frac{v^n(x_i - a_1\Delta t, y_j - a_2\Delta t) - v^n(x_i, y_j)}{\Delta t} + H^*(a)\right] \quad (2.9)$$

and, finally, the time explicit scheme

$$v^{n+1}_{i,j} = \min_{a \in \mathbb{R}^2}\left[v^n(x_i - a_1\Delta t, y_j - a_2\Delta t) + \Delta t H^*(a)\right]. \quad (2.10)$$

It is clear from (2.10) that the SL-schemes have the same structure of the representation formula of the exact solution written for $v_0 = v^n$ and $t = \Delta t$. However, several steps are necessary in order to compute the solution. The first is to compute the value of v on the right-hand side by an interpolation procedure based on the values on the nodes of the lattice L. Then, one has to determine $H^*(a)$ so that we can finally compute the minimum for $a \in \mathbb{R}^2$. Let us examine the difficulties at every step of this procedure.

The reconstruction step is not a major difficulty. One can use many types of procedures which allow to compute the value of v at the foot of the characteristics by the values on the nodes of the lattice. High-order polynomial interpolations will introduce oscillations if the solution is not regular, however, viscosity solutions are typically Lipschitz continuous provided v_0 is Lipschitz continuous. At this stage, other accurate interpolation procedures can also be used, e.g., ENO or WENO interpolations. It is interesting to note that whenever H^* is known and the solution is a polynomial, the above scheme can give an approximate solution which is accurate to any order for arbitrary choices of Δt. If this is the case, the error only depends on the interpolation step and can be canceled just choosing an interpolation of the same order of the polynomial. The location of the foot of characteristics requires additional work with respect to finite differences schemes. It is easy and not expensive on structured grids when one simply divides the coordinate x (respectively y) by the space discretization step Δx (respectively Δy) in order to determine the cell containing the foot. This step can be more expensive on unstructured grids, particularly when the size of the triangles is not homogeneous.

A major difficulty when applying (2.10) is computing H^*. Sometimes it is possible (see the examples below) to determine its explicit expression, but in general one has to rely on its

approximation by the fast Legendre transform developed by Brenier [8] and Corrias [11]. This solution is feasible as far as the state space has two or three dimensions.

The last difficulty is that the minimum should be computed on an unbounded set (in principle, on the whole space). Despite Definition (2.10), the search for a minimum can be reduced to a bounded set in many cases. In particular, we will show later in this section that if H is Lipschitz continuous we can determine precise bounds for the compact set containing the minimum point.

Let us examine some interesting cases where H^* is known. If $H(p) = |p|$, the assumption (2.1) is not satisfied and $H^*(p)$ is not a real value for every p. In fact, it is easy to prove that

$$H^*(p) = \begin{cases} 0, & \text{for } |p| \le 1 \\ +\infty, & \text{elsewhere.} \end{cases} \tag{2.11}$$

By the above definition and (2.10), we can reduce the search for the minimum to the unit ball obtaining the following scheme

$$v_{i,j}^{n+1} = \min_{a \in \mathcal{B}(0,1)} v^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t). \tag{2.12}$$

If $H(p) = |p|^2/2$, the assumption (2.1) is satisfied and $H^*(p)$ is real valued for every p. In fact, it is easy to prove that $H^* = H$. Then, the scheme is explicit in time

$$v_{i,j}^{n+1} = \min_{a \in \mathbb{R}^2} \left[v^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) + \frac{|a|^2}{2} \right]. \tag{2.13}$$

In this case we still have to compute the minimum over an unbounded domain. The following lemma shows how to determine an equivalent compact set containing the minimum point. Its proof is analogous to the proof given in [12] for the one-dimensional case.

LEMMA 2.2. *Let $H : \mathbb{R}^N \to \mathbb{R}$ be continuous and convex. Moreover, let H satisfy* (2.1). *Then, there exists a compact set $A \subset \mathbb{R}^N$ such that*

$$H(p) = \sup_{a \in \mathbb{R}^N} \{p \cdot a - H^*(a)\} = \sup_{a \in A} \{p \cdot a - H^*(a)\}. \tag{2.14}$$

Proof. It suffices to prove that there exists a real constant K such that the set

$$A_K \equiv \{a \in \mathbb{R}^N : a \cdot p - H^*(a) \ge K\} \tag{2.15}$$

is not empty and bounded. Let us assume that $|p| \le M_p$, then

$$\frac{K + H^*(a)}{|a|} \le \frac{a}{|a|} \cdot p, \quad \forall a \in A_K, \tag{2.16}$$

which implies

$$\frac{K + H^*(a)}{|a|} \le |p| \le M_p, \quad \forall a \in A_K. \tag{2.17}$$

Since H satisfies (2.1), A_K must be bounded. Moreover, A_K is nonempty since it contains $\bar{a}$ for $K = K(\bar{a}) = -|\bar{a}| M_p - H^*(\bar{a})$. ■

The argument in the above proof shows that for $|p| \le M_p$ the search for the minimum related to the computation of $H(p)$ can be restricted to the set

$$A \equiv \{a \in I\!R^N : aM_p - H^*(p) \ge K(0) = -H^*(0)\}. \tag{2.18}$$

For $H(p) = |p|^2/2$, this gives the inequality

$$|a|M_p - \frac{|a|^2}{2} \ge 0,$$

which implies

$$A = \{a : |a| \le 2M_p\}. \tag{2.19}$$

Using similar arguments, it can also be proved that if H is Lipschitz continuous with constant L_H, $H^*(p) = +\infty$ for $|p| > L_H$. This naturally implies that the search for the minimum can be restricted to the set $A = \{a : |a| \le L_H\}$. In our implementation the minimum is first approximately located by a tabulation over a finite number of points, then precisely computed by a Powell-type algorithm (see [7] and the NETLIB routine PRAXIS). Constraints are treated by penalization.

We conclude this section with some remarks and extensions. Let us observe first that the same approach can be followed to deal with more general Hamiltonians of the form $H(x, \nabla u)$. In fact, if H is continuous with respect to $(x, \nabla u)$, convex in ∇u, and satisfies (2.1) uniformly in x then the relation with a finite horizon control problem is still valid and the viscosity solution of the corresponding Hamilton–Jacobi equation is still the value function. The only difference is that the running cost is now depending on x, i.e., $f(x, a) = H^*(x, a)$, and this implies that characteristics are no longer straight lines.

More general representation formulae for convex Hamiltonians $H(v, \nabla v)$ can be found in [1, 5]. For nonconvex Hamiltonians the representation formula for the solution (to be always understood in the viscosity sense) relies on the interpretation in terms of differential games. The Hamiltonian can be written as a min–max on two parameters and the solution is the value function of the game. The interested reader will find in [15] and [2] some representation formulas for nonconvex hamiltonians, whereas the numerical approximation of non-convex Hamilton–Jacobi equations related to pursuit–evasion games can be found in the survey paper [4] (see also Remark 5.1, p. 569 in [12]).

3. EQUIVALENCE WITH GODUNOV SCHEME

We will show in this section that the SL scheme is actually a generalization of the classical Godunov scheme for conservation laws. For this comparison to be significant, we will restrict to the situation of small Courant numbers (so that the Godunov scheme is stable) and use P_1 reconstructions in the SL scheme (so that it can appear as the integration of the piecewise constant reconstruction in the Godunov scheme).

Consider therefore the problem

$$\begin{aligned} u_t + H(u)_x &= 0 \\ u(x, 0) &= u_0(x), \end{aligned} \tag{3.1}$$

with $x \in \mathbb{R},\ t \geq 0,\ H(\cdot)$ smooth and convex, and u_0 an L^∞, compactly supported function on $\mathbb{R}$. Consider also the associated HJ equation

$$\begin{aligned} v_t + H(v_x) &= 0 \\ v(x,0) = v_0(x) &= \int_{-\infty}^{x} u_0(\xi)\, d\xi. \end{aligned} \tag{3.2}$$

It is well known that the viscosity solution of (3.2) is the integral of the entropic solution of (3.1) for any $t \geq 0$, i.e.,

$$v(x,t) = \int_{-\infty}^{x} u(\xi,t)\, d\xi. \tag{3.3}$$

Consider now the behavior of the classical Godunov scheme for (3.1) and of the semi-Lagrangian scheme for (3.2), assuming that the Courant number is small enough to make the first one stable. Defining the (piecewise constant) reconstruction operator

$$\Pi_0[u](x) = \frac{1}{\Delta x} \int_{x_{j-1}}^{x_j} u(\xi, t_n)\, d\xi \quad \text{for any } x \in [x_{j-1}, x_j) \tag{3.4}$$

and $\bar{u}$ as the solution for $t \geq t_n$ of the initial value problem

$$\begin{aligned} \bar{u}_t + H(\bar{u})_x &= 0 \\ \bar{u}(t_n) &= u_\Delta(t_n) \end{aligned} \tag{3.5}$$

we have

$$u_\Delta(x, t_{n+1}) = \Pi_0[\bar{u}(t_{n+1})](x). \tag{3.6}$$

On the other hand, defining

$$\Pi_1[v](x) = v(x_{j-1}) + \frac{v(x_j) - v(x_{j-1})}{\Delta x}(x - x_{j-1})(x \in [x_{j-1}, x_j)) \tag{3.7}$$

and $\bar{v}$ as the solution of

$$\begin{aligned} \bar{v}_t + H(\bar{v}_x) &= 0 \\ \bar{v}(t_n) &= v_\Delta(t_n), \end{aligned} \tag{3.8}$$

then (using the Lax–Hopf representation formula) the numerical solution v_Δ of the SL scheme for (3.2) is

$$v_\Delta(x, t_{n+1}) = \Pi_1[\bar{v}(t_{n+1})](x). \tag{3.9}$$

If we assume that at time $t_n = n\Delta t$ the numerical solutions $u_\Delta,\ v_\Delta$ satisfy

$$v_\Delta(x, t_n) = \int_{-\infty}^{x} u_\Delta(\xi, t_n)\, d\xi, \tag{3.10}$$

by (3.3) and the definition of $\bar{u}$, $\bar{v}$ we obtain

$$\bar{v}(x, t_{n+1}) = \int_{-\infty}^{x} \bar{u}(\xi, t_{n+1})\, d\xi, \tag{3.11}$$

and also, using the definition of Π_1,

$$\begin{aligned} v_\Delta(x_j, t_{n+1}) &= \Pi_1[\bar{v}(t_{n+1})](x_j) = \bar{v}(x_j, t_{n+1}) \\ &= \int_{-\infty}^{x_j} \bar{u}(\xi, t_{n+1})\, d\xi \\ &= \sum_{k=-\infty}^{j} \int_{x_{k-1}}^{x_k} \bar{u}(\xi, t_{n+1})\, d\xi \\ &= \sum_{k=-\infty}^{j} \int_{x_{k-1}}^{x_k} u_\Delta(\xi, t_{n+1})\, d\xi, \end{aligned}$$

where the last equality is motivated by the definition of Π_0. We obtain therefore

$$v_\Delta(x_j, t_{n+1}) = \int_{-\infty}^{x_j} u_\Delta(\xi, t_{n+1})\, d\xi, \tag{3.12}$$

which shows (taking into account that Π_0 is piecewise constant and Π_1 piecewise linear) that (3.10) also holds at time t_{n+1}. Then, we have proved the equivalence of the two schemes as stated by the following.

THEOREM 3.1. *Let $H : \mathbb{R} \to \mathbb{R}$ be continuous and convex and let H satisfy* (2.1). *Let us denote by u_Δ the approximate solution of* (3.1) *corresponding to the Godunov method and by v_Δ the approximate solution of* (3.2) *corresponding to the SL-scheme with piecewise linear (P_1) reconstruction. Then, for $\Delta t/\Delta x$ sufficiently small,* (3.12) *holds true.*

4. GENERAL PROPERTIES

Although the theory of semi-Lagrangian approximation for nonlinear problems is still incomplete, the schemes described in the preceding sections have some interesting properties which we will briefly describe here. For simplicity, we will give the proofs in $\mathbb{R}^1$.

Let us start from the local truncation error. We have

$$\begin{aligned} \left|v(x_i, t_{n+1}) - v_i^n\right| = \Big|&\min_a [v(x_i - a\Delta t, t_n) + \Delta t H^*(a)] \\ &- \min_a [v^n(x_i - a\Delta t) + \Delta t H^*(a)]\Big|. \end{aligned} \tag{4.1}$$

Denoting by $\bar{a}_1$, $\bar{a}_2$, respectively, the argmin of first and of the second term of the right-hand side of (4.1), we obtain

$$\begin{aligned} \left|v(x_i, t_{n+1}) - v_i^n\right| \le \max\{&|v(x_i - \bar{a}_1\Delta t, t_n) - v^n(x_i - \bar{a}_1\Delta t)|, \\ &|v(x_i - \bar{a}_2\Delta t, t_n) - v^n(x_i - \bar{a}_2\Delta t)|\}. \end{aligned} \tag{4.2}$$

Assuming that the solution v is regular and $v_i^n = v(x_i, t_n)$, for any i, (4.2) implies

$$E_{\Delta t} \equiv \frac{1}{\Delta t}\left|v(x_i, t_{n+1}) - v_i^n\right| \le \frac{O(\Delta x^p)}{\Delta t}, \tag{4.3}$$

where the p is the order of of accuracy of the space recontruction used to compute $v^n(x_i - a\Delta t)$. Note that the above estimate has just one term because our Hamiltonian depends only on ∇u. For more general Hamiltonians depending on $(x, \nabla u)$, the local truncation error will have a second term which takes into account the error in the approximation of characteristic lines. Such a term can be obtained following the arguments in [19] and it turns out to be $O(\Delta t^q)$, where q is the order of the discrete approximation used for characteristics.

As far as stability is concerned, we just recall that the schemes corresponding to a P_1 space reconstruction are monotone and unconditionally stable since

$$\|v^{n+1}\|_{1,\infty} \le (1 + C\Delta t)\|v^n\|_{1,\infty}, \tag{4.4}$$

with $C = 0$. For high-order reconstructions, one can prove that the same estimate holds true with a positive C under the restrictive assumption $\Delta x = O(\Delta t^3)$ (see [23]).

Once proved that the scheme is consistent, we can check that for the low-order (P_1, Q_1) implementations of the scheme the convergence theory of Lin and Tadmor (see [29]) applies since numerical solutions are uniformly semiconcave. Assume for simplicity that $N = 2$, and suppose moreover that the grid is orthogonal and uniform with step $\Delta x = \Delta y$, and that at the n-th step the discrete semiconcavity assumption

$$v_{r-h,s-k}^n - 2v_{r,s}^n + v_{r+h,s+k}^n \le C\Delta x^2 \tag{4.5}$$

holds for the discrete solution at a node x_{rs} with an increment $\pm(h\Delta x, k\Delta x)$ (h, k integers). Then, at the $(n+1)$-th step we have

$$\begin{aligned}
&v_{i-h,j-k}^{n+1} - 2v_{i,j}^{n+1} + v_{i+h,j+k}^{n+1} \\
&\quad = \min_a \left[\Delta t H^*(a) + v^n(x_{i-h,j-k} + a\Delta t)\right] - 2\min_a\left[\Delta t H^*(a)\right. \\
&\qquad \left. + v^n(x_{ij} + a\Delta t)\right] + \min_a[\Delta t H^*(a) + v^n(x_{i+h,j+k} + a\Delta t)] \\
&\quad \le \Delta t H^*(\bar a) + v^n(x_{i-h,j-k} + \bar a\Delta t) - 2\Delta t H^*(\bar a) - 2v^n(x_{ij} + \bar a\Delta t) \\
&\qquad + \Delta t H^*(\bar a) + v^n(x_{i+h,j+k} + \bar a\Delta t),
\end{aligned} \tag{4.6}$$

where we have bounded the sum from above using as $\bar a$ the minimizer for the node x_{ij}. Setting now $v^n(x_{ij} + \bar a\Delta t) = \sum_{l,m}\ \lambda_{lm}\, v_{l,m}^n$ and recalling that the P_1, Q_1 reconstructions consist of a convex combination of the values in neighboring nodes, we have that $\lambda_{lm} \ge 0$ and $\sum_{l,m} \lambda_{lm} = 1$. By periodicity of the grid we obtain therefore

$$\begin{aligned}
&v_{i-h,j-k}^{n+1} - 2v_{i,j}^{n+1} + v_{i+h,j+k}^{n+1} \\
&\quad \le \sum_{l,m} \lambda_{lm} v_{l-h,m-k}^n - 2\sum_{l,m} \lambda_{lm} v_{l,m}^n + \sum_{l,m} \lambda_{lm} v_{l+h,m+k}^n \\
&\quad = \sum_{l,m} \lambda_{lm}\left(v_{l-h,m-k}^n - 2v_{l,m}^n + v_{l+h,m+k}^n\right) \le \sum_{l,m} \lambda_{lm} C\Delta x^2 = C\Delta x^2.
\end{aligned} \tag{4.7}$$

Lastly, the inequality can be extended to any increment in the form considered in [29] by convex combination.

The semiconcavity bound has been proved in [12] (see Theorem 5.2) for the one dimensional P_1-scheme. It is worth noting that semiconcavity plays an important role in this problem. In the exact equation, if the initial solution v_0 is semiconcave and H^* is regular, then the minimization which appears in (2.6) would always be performed on a semiconcave function resulting from the sum of two terms, one smooth and convex and the second itself semiconcave. The evaluation of the solution v (whose discrete counterpart is the reconstruction step) is performed at the minimum point where the solution is at least differentiable (see [3]). Although we are only able to prove uniform discrete semiconcavity for low-order reconstructions, numerical tests show that this property also holds, at least approximately, in the high-order case. Then it is not surprising that the scheme may take advantage from the uniform semiconcavity of the solution as we will show in some numerical tests.

5. NUMERICAL TESTS

Test 1. Eikonal Propagation of Fronts

The first test refers to the HJ equation

$$\begin{aligned} v_t(x,t) + f(x)\cdot\nabla v(x,t) + |\nabla v(x,t)| = 0 \\ v(x,0) = v_0(x), \end{aligned} \tag{5.1}$$

where the advecting vectorfield f has streamlines which rotate anticlockwise around the origin.

The problem is considered in $[-2,2]^2$ with a 50×50 grid. It is well known that in this case the motion of a level curve of the viscosity solution results from the superposition of an evolution at constant speed, and a passive advection driven by the field f. The initial data v_0 are chosen so as to generate the level set shown in the first picture of Fig. 1. The following pictures show the evolution of the front, and we remark that despite the small number of nodes, the evolution is remarkably isotropic, and singularities generated by front collapsing do not introduce significant instabilities.

Test 2. Strictly Convex Hamiltonian

This test refers to the HJ equation:

$$\begin{aligned} v_t(x,t) + \frac{1}{2}|\nabla v(x,t)|^2 = 0 \\ v(x,0) = v_0(x) = \max(0, 1-|x|^2). \end{aligned} \tag{5.2}$$

The problem is considered in $[-2,2]^2$. The solution of this test problem generates a singularity in the gradient for $t > 1/2$.

The exact solution of (5.2) can be explicitly computed and for $t > 1/2$ it reads

$$v(t,x) = \begin{cases} \frac{(|x|-1)^2}{2t} & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

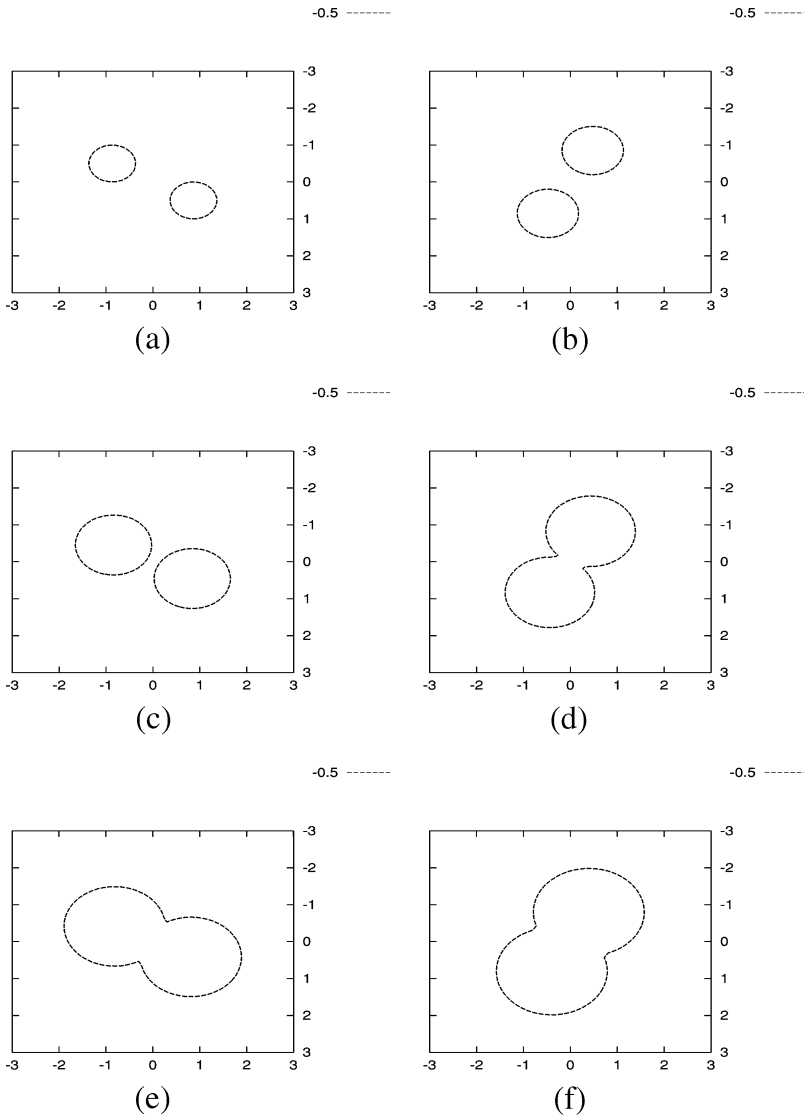

FIG. 1. Approximate solutions (level curves) for Test 1.

The solution, computed with $\Delta t = 0.1$ and a cubic reconstruction, is shown in Fig. 2 every two time steps between $t = 0$ and $t = 1$. Table I shows L^∞ and local errors at $T = 1$ (the latter being computed in a suitable smooth portion of the solution) for different numbers of nodes on the edge of the computational domain. The error table shows that the solution is computed with good accuracy even with a low number of nodes, although the pointwise numerical error does not improve in smooth regions (but see the discussion of this point after test 3). Again, we point out that the singularity in the gradient is well resolved, and that the scheme does not introduce oscillations or other instabilities.

TABLE I
Numerical Errors for Test 2

Nodes	L^∞ error	L^1 error	local error
25	5.75×10^{-2}	6.57×10^{-2}	5.75×10^{-2}
50	1.85×10^{-2}	2.76×10^{-2}	1.66×10^{-2}
100	6.17×10^{-3}	8.32×10^{-3}	6.17×10^{-3}

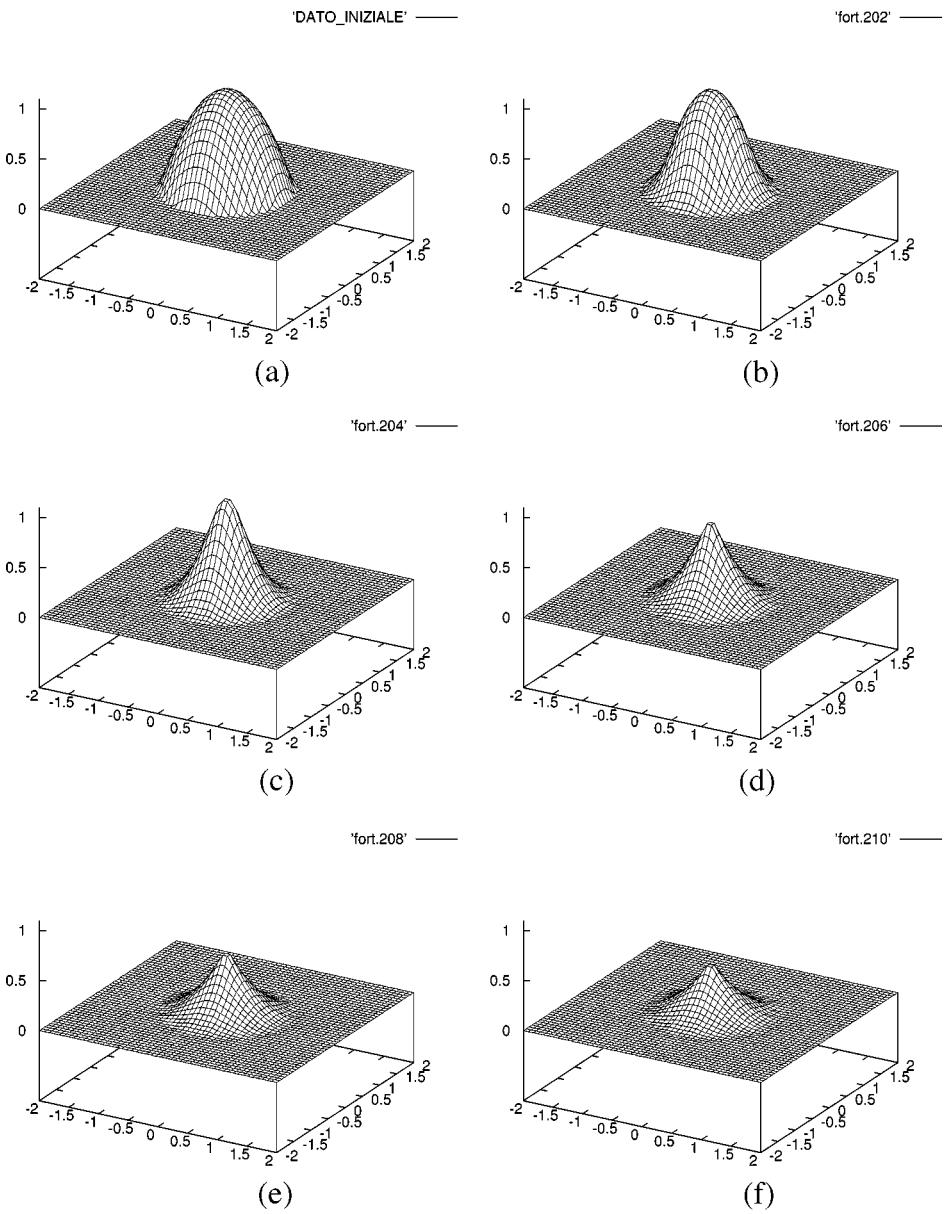

FIG. 2. Approximate solutions for Test 2.

Test 3. Strictly Convex Hamiltonian with Semiconcave Initial Data

Let us consider the problem

$$\begin{aligned} v_t(x,t) + \frac{1}{2}|\nabla v(x,t)|^2 &= 0 \\ v(x,0) = v_0(x) &= \max(0, |x|^2 - 1). \end{aligned} \tag{5.3}$$

The problem is considered again in $[-2, 2]^2$. The exact solution of this test problem reads

$$v(t,x) = \begin{cases} \frac{|x|^2}{2t+1} - 1 & \text{if } |x| \le \sqrt{2t+1} \\ 0 & \text{if } |x| \ge \sqrt{2t+1}. \end{cases}$$

Figure 3 shows the numerical solution computed for $t \in [0, 1]$ with $\Delta t = 0.1$ and a cubic reconstruction, and Table II shows L^∞, L^1, and local errors at $T = 1$. In this case, the L^∞ error has the same order of magnitude as before, whereas both L^1 and local error have a very strong improvement.

The comparison of numerical errors in tests 2 and 3 shows an interesting feature of the scheme. The uniform semiconcavity of v, along with the assumptions on H, heuristically shows that the minimization step in the scheme (2.10) would be performed on a semiconcave function, so that the "upwind" point $x_j + \int_0^{\Delta t} \alpha(s)\, ds$ in which the reconstruction is performed (which corresponds to the argmin in the scheme) can only be a regular point for v. This results in a faster convergence in regions away from singularities, as the results of tests 2 and 3 show.

Test 4. A Front Evolving on a Manifold

In the last test we consider the HJ equation,

$$\begin{aligned} v_t(x,t) + H(\nabla v(x,t)) = 0, \quad x \in M \\ v(x,0) = v_0(x), \end{aligned} \tag{5.4}$$

posed on a Riemannian manifold M, so that the symbol ∇ should now be understood as the intrinsic gradient. In particular, the manifold is a torus in this test and we set $H(p) = |p|$, so that a level curve would propagate at constant speed along geodesics.

The manifold is mapped on $[-\pi, \pi]^2$ with doubly periodic boundary conditions, and Eq. (5.4) is rewritten in the planar coordinates ξ as

$$v(\xi,t) + \sup_\beta [-\beta \times \nabla v(\xi,t) - H^*(J^T(\xi)\,\beta)] = 0, \quad \xi, \beta \in \mathbb{R}^2, \tag{5.5}$$

where again H^* is the Legendre transform of H, and J is the Jacobian matrix of the transformation mapping $[-\pi, \pi]^2$ into the torus of $\mathbb{R}^3$. The form (5.5) allows to treat the Riemannian case as a straightforward adaptation of the euclidean case, whenever the parametrization of the manifold is known. The level sets of the numerical solution are shown in Fig. 4.

TABLE II
Numerical Errors for Test 3

Nodes	L^∞ error	L^1 error	local error
25	1.57×10^{-2}	2.88×10^{-2}	2.89×10^{-6}
50	4.03×10^{-3}	8.42×10^{-4}	7.36×10^{-14}
100	6.46×10^{-4}	3.29×10^{-5}	1.76×10^{-13}

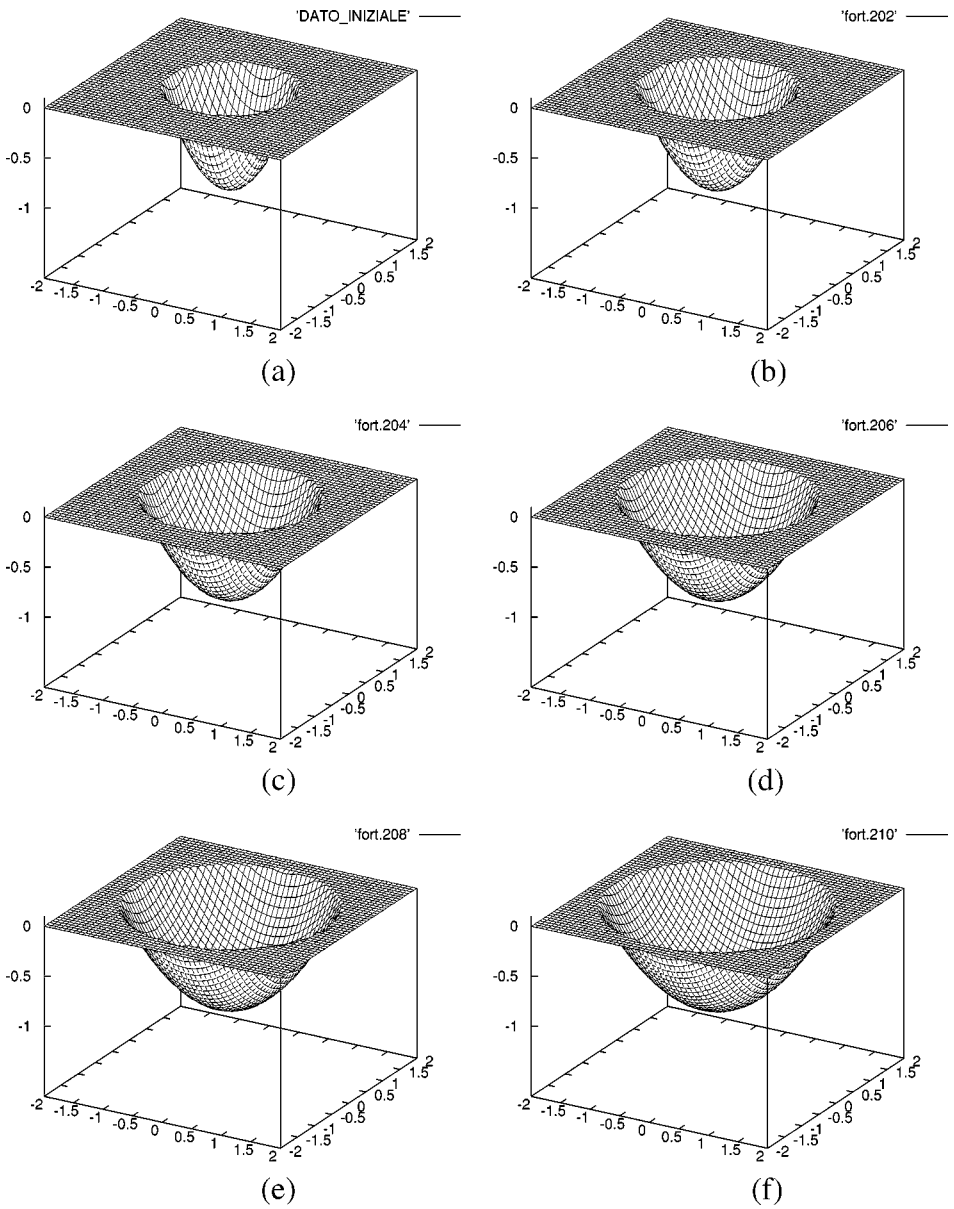

FIG. 3. Approximate solutions for Test 3.

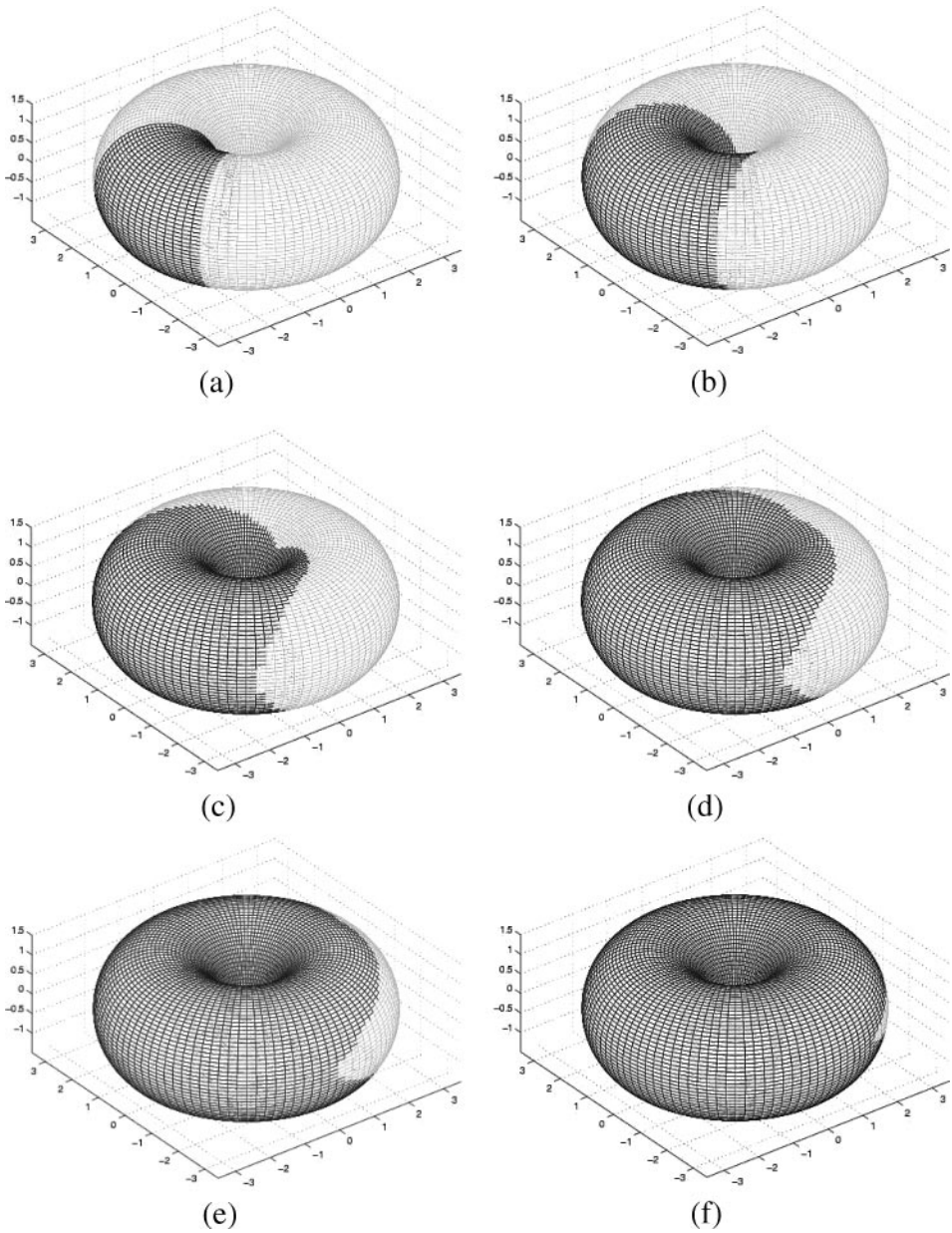

FIG. 4. Approximate solutions (level curves) for Test 4.

REFERENCES

1. M. Bardi and L. C. Evans, On Hopf's formulas for solutions of Hamilton–Jacobi equations, *Nonlinear Anal.* **8**, 1373 (1984).
2. M. Bardi and S. Faggian, Hopf-type estimates and formulas for non-convex non-concave Hamilton–Jacobi equations, *SIAM J. Math. Anal.* **29**, 1067 (1998).
3. M. Bardi and I. Capuzzo Dolcetta, Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations (Birkhäuser, Basel, 1997).

4. M. Bardi, M. Falcone, and P. Soravia, Numerical methods for pursuit–evasion games via viscosity solutions, in *Stochastic and Differential Games: Theory and Numerical Methods* edited by M. Bardi, T. Parthasarathy, and T. E. S. Raghavan (Birkhäuser, Basel, 1999).
5. M. Bardi and S. Osher, The nonconvex multidimensional Riemann problem for Hamilton–Jacobi equations, *SIAM J. Math. Anal.* **22**, 344 (1991).
6. G. Barles, Solutions de viscositè des equations d'Hamilton–Jacobi (Springer-Verlag, Berlin/New York, 1998).
7. R. Brent, Algorithms for minimization without derivatives (Prentice-Hall, Englewood Cliffs, 1973).
8. Y. Brenier, Un algorithme rapide pour le calcul de trasformées de Legendre–Fenchel discretes, *C.R. Acad. Sci. Paris Ser. I Math.* **308**, 587 (1989).
9. M. Briani and M. Falcone, A semi-lagrangian scheme for advection–diffusion equations. Preprint (2000).
10. F. Camilli and M. Falcone, An approximation scheme for the optimal control of diffusion processes, *Math. Model. Numer. Anal.* **29**, 97 (1995).
11. L. Corrias, Fast Legendre-Fenchel transform and applications to Hamilton–Jacobi equations and conservation laws, *SIAM J. Numer. Anal.* **33**, 1534 (1996).
12. L. Corrias, M. Falcone, and R. Natalini, Numerical schemes for conservation laws via Hamilton–Jacobi equations, *Math. Comp.* **64**, 555 (1995).
13. R. Courant, E. Isaacson, and M. Rees, On the solution of nonlinear hyperbolic differential equations by finite differences, *Comm. Pure Appl. Math.* **5**, 243 (1952).
14. M. G. Crandall and P. L. Lions, Two approximations of solutions of Hamilton–Jacobi equations, *Math. Comput.* **43**, 1 (1984).
15. L. C. Evans and P. E. Souganidis, Differential games and representation formulas for solutions of Hamilton–Jacobi equations, *Indiana Univ. Math. J.* **33**, 773 (1984).
16. M. Falcone, A numerical approach to the infinite horizon problem of deterministic control theory, *Appl. Math. Optim.* **15**, 1 (1987). *Corrigenda in Appl. Math. Optim*. **23**, 213 (1991).
17. M. Falcone, The minimum time problem and its applications to front propagation, in *Motion by Mean Curvature and Related Topic* (Trento, 1992, pp. 70–88; de Gruyter, Berlin, 1994).
18. M. Falcone, Numerical solution of dynamic programming equations. In *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations, App. A* (Birkhäuser, Basel, 1997).
19. M. Falcone, R. Ferretti, Discrete-time high-order schemes for viscosity solutions of Hamilton-Jacobi equations. *Numer. Math.* **67**, 315 (1994).
20. M. Falcone, R. Ferretti, Convergence analysis for a class of semi-lagrangian advection schemes. *SIAM J. Numer. Anal.* **35**, 909 (1998).
21. M. Falcone, T. Giorgi, An approximation scheme for evolutive Hamilton–Jacobi equations, in *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming* edited by W. M. McEneaney, G. Yin, Q. Zhang (Birkhäuser, Basel, 1998).
22. R. Ferretti, A Technique for High-Order Treatment of Diffusion Terms in Semi-Lagrangian Schemes. Preprint (2000).
23. R. Ferretti, Equicontinuity of some large time-step approximations to convex Hamilton–Jacobi equations, in *Proceedings of the Workshop "HJB2000"* (INRIA, Paris, 2000).
24. E. Hopf, Generalized solutions of non linear equations of first order. *J. Math. Mech.* **14**, 951 (1965).
25. G. S. Jiang, D. Peng, Weighted ENO schemes for Hamilton–Jacobi equations. *SIAM J. Sci. Comput.* **21**, 2126 (2000).
26. S. N. Kružkov, The method of finite differences for a first-order nonlinear equation with many independent variables. *USSR Comput. Math. Math. Phys.* **6**, 136 (1966).
27. R. J. Leveque, *Numerical Methods for Conservation Laws* (Birkhäuser, Basel, 1992).
28. C. T. Lin, E. Tadmor, High-resolution nonoscillatory central schemes for Hamilton–Jacobi equations. *SIAM J. Sci. Comput.* **21**, 2163 (2000).
29. C. T. Lin and E. Tadmor, L^1 stability and error estimates for Hamilton–Jacobi solutions, *Numer. Math.*, submitted.

30. S. Osher, The Riemann problem for convex scalar conservation laws and Hamilton–Jacobi equations. *Proc. Amer. Math. Soc.* **89**, 641 (1983).
31. S. Osher and J. A. Sethian, Fronts propagating with curvature-dependent speed: algorithms based on Hamilton–Jacobi formulations. *J. Comput. Phys.* **79**, 12 (1988).
32. S. Osher and C. W. Shu, High-order essentially non-oscillatory schemes for Hamilton–Jacobi equation. *SIAM J. Numer. Anal.* **28**, 907 (1991).
33. J. A. Sethian, Level set method. In *Evolving Interfaces in Geometry, Fluid Mechanics, Computer Vision, and Materials Science.* Cambridge Monographs on Applied and Computational Mathematics (Cambridge Univ. Press, Cambridge, 1996), Vol. 3.
34. A. N. Staniforth and J. Côtè, Semi-Lagrangian integration schemes for atmospheric models—A review. *Mon. Wea. Rev.* **119**, 2206 (1991).
35. J. Strain, Semi-Lagrangian methods for level set equations. *J. Comput. Phys.* **151**, 498 (1999).
36. J. C. Strickwerda, *Finite Difference Schemes and Partial Differential Equations* (Pacific Grove, California, Wadsworth and Brooks/Cole, 1989).